# Anhui University

## Semester 1, 2004-2005 Final Examination (Paper A)

### Model Answer and Referee Criterion for *Numerical Analysis*

In question 1-6, please choose the correct answer (only one is correct)
**1.** (5 marks) Word "MATLAB" comes form
(A) Mathematics Laboratory
(B) Matrix Laboratory
(C) Mathematica Laboratory
(D) Maple Laboratory
*Answer.* (B)

**2.** (5 marks) The matrix

$$\begin{pmatrix} 4 & -1 & 1 \\ 4 & -8 & 1 \\ -2 & 1 & 5 \end{pmatrix}$$

is
(A) a strictly diagonally dominant matrix;
(B) not a strictly diagonally dominant matrix;
(C) a singular matrix;
(D) a matrix whose determinant is equal to zero.
*Answer.* (A)

**3.** (5 marks) The computational complexity of Gaussian elimination for solving linear equation systems $AX = B$ ($A$ is $N \times N$ matrix) is

(A) $O(N)$, (B) $O(N^2)$, (C) $O(N^3)$, (D) $O(N^4)$.

*Answer.* (C)

**4.** (5 marks) Assume that $f(x)$ is defined on $[a, b]$, which contains equally spaced nodes $x_k = x_0 + hk$. Additionally, assume that $f''(x)$ is continuous on $[a, b]$. If we use Lagrange interpolation polynomial

$$P_1(x) = \sum_{k=0}^{1} f(x_k) L_{1,k}(x)$$

to approximate $f(x)$, then the error $E_1(x)$ is

(A) $O(1)$ (B) $O(h)$ (C) $O(h^2)$ (D) $O(h^3)$

*Answer.* (C)

**5.** (5 marks) Degree 4 Chebyshev Polynomial $T_4(x)$ is

| | | | |
|---|---|---|---|
| (A) | an odd function, | (B) | an even function |
| (C) | not odd or even function, | (D) | both odd and even function |

*Answer.* (B)

**6.** (5 marks) Padé Approximation is

(A) A rational polynomial approximation
(B) A polynomial approximation
(C) A triangular polynomial approximation
(D) A linear function approximation

*Answer.* (A)

**7.** (15 marks) Use the false position method to find the root of $x\sin(x)-1=0$ that is located in the interval $[0,2]$ (the function $\sin(x)$ is evaluated in radians).
*Solution.* Starting with $a_0=0$ and $b_0=2$, we have $f(0)=-1$ and $f(2)=0.8186$, so a root lies in the interval $[0,2]$.

$\cdots$ 2 marks

Using formula

$$c_n = b_n - \frac{f(b_n)(b_n-a_n)}{f(b_n)-f(a_n)},$$

$\cdots$ 7 marks

we get

$$c_0 = 2 - \frac{0.8186(2-0)}{0.8186-(-1)} = 1.0997 \quad \text{and} \quad f(c_0) = -0.0200$$

The function changes sign on the interval $[c_0, b_0] = [1.0997, 2]$, so we squeeze from the left and set $a_1 = c_0$ and $b_1 = b_0$. The formula produces the next approximation:

$$c_1 = 2 - \frac{0.8186(2-1.0997)}{0.8186-(-0.0200)} = 1.1212$$

and

$$f(c_1) = 0.0098.$$

$\cdots$ 11 marks

Next $f(x)$ changes sign on $[a_1, c_1] = [1.0997, 1.1212]$, and the next decision is to squeefrom the right and set $a_2 = a_1$ and $b_2 = c_1$. Hence we get $c_2 = 1.1141$ and

$$f(c_2) = 0.0000$$

$\cdots$ 15 marks

**8.** (15 marks) In the following linear equation systems

$$\begin{aligned} 4x - y + z &= 7 \\ 4x - 8y + z &= -21 \\ -2x + y - 5z &= 15, \end{aligned}$$

start with $\boldsymbol{P}_0 = (1, 2, 2)$, and use Gauss-Seidel iteration to find $\boldsymbol{P}_k$ for $k = 1, 2$. Will Gauss-Seidel iteration converge to the solution?
*Solution.* The Gauss-Seidel iteration is

$$x_{k+1} = \frac{7 + y_k - z_k}{4}$$

$$y_{k+1} = \frac{21 + 4x_{k+1} + z_k}{8}$$

$$z_{k+1} = \frac{15 + 2x_{k+1} - y_{k+1}}{5}$$

$\cdots$ 7 marks

Substitute $y_0 = 2$ and $z_0 = 2$ into the first equation of the systems above and obtain

$$x_1 = \frac{7 + 2 - 2}{4} = 1.75.$$

Then substitute $x_1 = 1.75$ and $z_0 = 2$ into the second equation and get

$$y_1 = \frac{21 + 4(1.75) + 2}{8} = 3.75.$$

Finally, substitute $x_1 = 1.75$ and $y_1 = 3.75$ into the third equation to get

$$z_1 = \frac{15 + 2(1.75) - 3.75}{5} = 2.95.$$

Using the same reason, we get

$$x_2 = 1.95, \quad y_2 = 3.96, \quad \text{and} \quad z_2 = 2.98$$

$\cdots$ 13 marks

Since the exact solution of the linear equation systems is $\boldsymbol{P} = (2, 4, 3)$, Gauss-Seidel iteration $\boldsymbol{P}_k$ will converge to the solution.

$\cdots$ 15 marks

**9.** (15 marks) Consider $y = f(x) = \cos(x)$ over $[0.0, 1.2]$. Use the three nodes $x_0 = 0.0, x_1 = 0.6$, and $x_2 = 1.2$ to construct a quadratic interpolation polynomial $P_2(x)$.
*Solution.* The quadratic Lagrange interpolation polynomial for $f(x)$ is

$$P_2(x) = \sum_{k=0}^{2} y_k L_{2,k}(x)$$

$$= y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$\cdots$ 7 marks

Using $x_0 = 0.0, x_1 = 0.6, x_2 = 1.2$ and $y_0 = \cos(0.0) = 1, y_1 = \cos(0.6) = 0.8253$, and $y_2 = \cos(1.2) = 0.3623$ in the formula above produces

$$P_2(x) = 1.0 \frac{(x - 0.6)(x - 1.2)}{(0.0 - 0.6)(0.0 - 1.2)} + 0.8253 \frac{(x - 0.0)(x - 1.2)}{(0.6 - 0.0)(0.6 - 1.2)}$$

$$+0.3623\frac{(x-0.0)(x-0.6)}{(1.2-0.0)(1.2-0.6)}$$
$$= 1.3888(x-0.6)(x-1.2) - 2.2925(x-0.0)(x-1.2)$$
$$+0.5032(x-0.0)(x-0.6).$$

$\cdots$ 15 marks

**10.** (15 marks) Consider $f(x) = 2 + \sin(2\sqrt{x})$. Use the composite Simpson rule with 11 sample points to compute an approximation to the integral of $f(x)$ taken over $[1, 6]$.

*Solution.* To generate 11 sample points, we must use $M = 5$ and $h = (6 - 1)/10 = 1/2$. Using formula

$$S(f,h) = \frac{h}{3}(f(a)+f(b)) + \frac{2h}{3}\sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3}\sum_{k=1}^{M} f(x_{2k-1}),$$

$\cdots$ 7 marks

the computation is

$$S(f,\frac{1}{2}) = \frac{1}{6}(f(1)+f(6)) + \frac{1}{3}(f(2)+f(3)+f(4)+f(5))$$
$$+\frac{2}{3}(f(\frac{3}{2}) + f(\frac{5}{2}) + f(\frac{7}{2} + f(\frac{9}{2}) + (\frac{11}{2})$$
$$= \frac{1}{6}(2.9092 + 1.0173)$$
$$+\frac{1}{3}(2.3080 + 1.6830 + 1.2431 + 1.0287)$$
$$+\frac{2}{3}(2.6381 + 1.9793) + 1.4353 + 1.1083 + 1.0002)$$
$$= \frac{1}{6}(3.9166+) + \frac{1}{3}(6.2630) + \frac{2}{3}(8.1613)$$
$$= 0.6544 + 2.0876 + 5.4408 = 8.1830.$$

$\cdots$ 15 marks

**11.** (10 marks) Assume that $g \in C[a,b]$. If the range of the mapping $y = g(x)$ satisfies $y \in [a,b]$ for all $x \in [a,b]$, then $g$ has a fixed point in $[a,b]$.

*Proof.* If $g(a) = a$ or $g(b) = b$, the assertion is true. Otherwise , the values of $g(a)$ and $g(b)$ must satisfy $g(a) \in (a,b]$ and $g(b) \in [a,b)$. The function $f(x) = x - g(x)$ has the property that

$$f(a) = a - g(a) < 0 \quad \text{and} \quad f(b) = b - g(b) > 0.$$

$\cdots$ 5 marks

Now apply **Zero Value Theorem** to $f(x)$, and conclude that there exists a number $P$ with $P \in (a,b)$ so that $f(P) = 0$. Therefore, $P = g(P)$ and $P$ is the desired fixed point of $g(x)$.

$\cdots$ 10 marks